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The proper L-S category of Whitehead manifolds

M. Cárdenas [a], F. Muro [b,*], A. Quintero [a]

[a] *Departamento de Geometría y Topología, Universidad de Sevilla, Spain*
[b] *Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany*

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Abstract

It is shown that the proper L-S category of an eventually end-irreducible, $\mathbb{R}^2$-irreducible Whitehead 3-manifold is 4. For this we prove, in the category of germs at infinity of proper maps, a partial analogue of the characterization by Eilenberg and Ganea of the L-S category of an aspherical space.

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1. Introduction

Ordinary homotopy invariants (numerical or functorial) are not strong enough to deal with non-compact spaces and more subtle invariants are needed to take into account the infinity behaviour of such spaces. For this one uses proper homotopy invariants. In particular, the proper analogue of the L-S category was introduced in [2]. Recall that the Lusternik–Schnirelmann (L-S) category of a space X, $cat(X)$, is the least number k such that there exists an open cover $\{U_1, \dots, U_k\}$ for which each inclusion $U_j \subset X$ is nullhomotopic in X. The homotopical significance of this number was pointed out by Borsuk and studied sys-

* Corresponding author.
E-mail addresses: muro@mpim-bonn.mpg.de (F. Muro), quintero@us.es (A. Quintero).

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tematically by Fox [14]; see [18] for a recent survey on L-S category and [10] for more details.

This paper continues the study of the proper L-S category carried out in a series of papers [2,3,8,9]. Here we work out the proper L-S category of a large family of open contractible 3-manifolds. Namely we prove the following theorem which shows a sharp contrast between the classical L-S category and its proper analogue for open manifolds.

Theorem A. *Let $W \neq \mathbb{R}^3$ be an eventually end-irreducible, $\mathbb{R}^2$-irreducible Whitehead 3-manifold. Then the proper L-S category of W is 4.*

Recall that a *Whitehead manifold* is a contractible and irreducible (i.e., any embedded sphere bounds a ball) open 3-manifold; see [27]. The first example of a such manifold different from $\mathbb{R}^3$ was given by J.H.C. Whitehead [29] and later McMillan [22] showed the existence of uncountably many manifolds of a similar nature. For all these manifolds Theorem A holds; see Remark 4.6. For the proof of Theorem A we first restate it in terms of proper homotopy invariants as Theorem 4.4 in Section 4 and then we obtain Theorem 4.4 as a consequence of the purely homotopical Theorem 5.6 proved in Section 5.

Theorem 5.6 is a partial analogue in the category of germs at infinity of proper maps of a well-known theorem due to Eilenberg and Ganea ([12] and [17]) which states that the ordinary L-S category of an aspherical space X is $\leqslant n$ if and only if the identity 1_X factorizes up to homotopy through an $(n-1)$-polyhedron. Theorem 5.6 requires that the fundamental pro-group *pro*-$\pi_1(X)$ of a properly aspherical space X is essentially monomorphic as well as the vanishing of the inverse limit $\varprojlim$ *pro*-$\pi_1(X)$. We show that the Whitehead manifolds in Theorem A satisfy these conditions. However there are simple examples of properly aspherical spaces (e.g., $X = S^1 \times \mathbb{R}^2$) for which *pro*-$\pi_1(X)$ is essentially monomorphic but $\varprojlim$ *pro*-$\pi_1(X) \neq 0$ and Theorem 5.6 does not hold.

A crucial step in the proof of Theorem 5.6 is the characterization of the proper L-S category of a 2-complex by its fundamental pro-group proved in [9]. In particular, for a Whitehead manifold W as in Theorem A we have $p\text{-}cat(W) = p\text{-}cat(W^2) + 1$. This equation holds for the ordinary L-S category of any closed 3-manifold M which is not a homotopy 3-sphere by the characterization ([17] or [24])

$$cat(M) = 3 \iff \pi_1(M) \text{ is free} \iff cat\big(M^2\big) = 2.$$

The corresponding equation and equivalences do not hold for arbitrary one-ended open 3-manifolds since $p\text{-}cat(M) = p\text{-}cat(M^2) = 3$ for $M = S^1 \times \mathbb{R}^2$. This paper intends to be a step further after [3] in the characterization of the proper L-S category of open 3-manifolds by their fundamental pro-group.

The definition of the proper L-S category will be given in the next section, which can be regarded as a continuation of this Introduction.

2. The proper L-S category

We work within the framework provided by the proper category $\mathcal{P}$ consisting of locally compact finite-dimensional polyhedra (or equivalently CW-complexes) and proper maps.

Recall that a *proper map* is a continuous map $f : X \to Y$ such that $f^{-1}(K)$ is compact for any compact subset $K \subset Y$. All maps and homotopies are assumed to be proper. We will use the diagram

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "F\Downarrow"] & Y \end{tikzcd}$$

to indicate that f and g are (properly) homotopic by F (also denoted by $F : f \Rightarrow g$).

In the category $\mathcal{P}$ the constant map $X \to \{p\}$ is not defined unless X is compact. Notwithstanding, the role of the point in $\mathcal{P}$ is played partially by the half-line $\mathbb{R}_+ = [0, \infty)$ since, for any space X in $\mathcal{P}$, there always exists a proper map $X \to \mathbb{R}_+$ which is unique up to proper homotopy; see 6.3.5 in [11].

A proper map $\alpha : \mathbb{R}_+ \to X$ is called a *ray* in X. A *properly based space* is a pair (X, α) with α a ray in X. This pair is *well-based* if α is a *proper cofibration*; that is, α satisfies the proper homotopy extension property (PHEP). We use the symbol "$\rightarrowtail$" for cofibrations. If (Y, β) is another properly based polyhedron we write $[X, Y]^{\mathbb{R}_+}$ for the set of proper homotopy classes relative to the base rays. The *proper wedge* of two properly well-based spaces $X \vee_{\mathbb{R}_+} Y$ is obtained by identifying their base rays. This wedge is properly well-based in the obvious way.

Given a space X in $\mathcal{P}$, a *system of* ∞*-neighbourhoods* of X is a decreasing sequence $\{W_j\}$ of subsets of X such that the closures of the complements $K_j = \overline{X - W_j}$ are compact subsets with $K_j \subset \operatorname{int} K_{j+1}$ and $X = \bigcup_{j=1}^{\infty} K_j$. A *Freudenthal end* of X is an element in the inverse limit $\mathcal{F}(X) = \varprojlim \pi_0(W_j)$. If $\mathcal{F}(X) = \{p\}$ then X is said to be *one-ended*. It is known that any polyhedron in $\mathcal{P}$ admits a system of ∞-neighbourhoods consisting of subpolyhedra; see [13].

A proper map $f : X \to Y$ is said to be *inessential* if there exists a diagram in $\mathcal{P}$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "r"'] & \arrow[d, phantom, "\Downarrow"] & Y \\ & * \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{1}$$

where "$*$" is either $\mathbb{R}_+$ or a one-point set. We can always assume that $* = \mathbb{R}_+$ if each component of Y is non-compact. Given a space X in $\mathcal{P}$ a (closed) subpolyhedron $i : A \subset X$ is said to be (properly) *inessential* if the inclusion i is an inessential map. The *proper L-S category* of X, $p\text{-}cat(X)$, is the least integer n such that X admits a cover $\{U_1, \ldots, U_n\}$ consisting of inessential subpolyhedra. For one-ended spaces the inequality

$$p\text{-}cat(X) \leqslant \dim X + 1$$

is easily checked.

Notice that the proper homotopy class of the map r in diagram (1) is unique. However, for $* = \mathbb{R}_+$ the proper homotopy class of the ray α depends on the set of homotopy classes $[\mathbb{R}_+, X]$. Each class $[\alpha] \in [\mathbb{R}_+, X]$ is called a *strong end* of X. For each strong end $[\alpha]$ one can also define the *proper L-S category of X based at* $[\alpha]$, $p\text{-}cat_{[\alpha]}(X)$, as the least integer n such that one finds a polyhedral cover $X = \bigcup_{j=1}^{n} U_j$ such that there are diagrams

$$\begin{tikzcd} U_j \arrow[rr, "i_j"] \arrow[dr, "r_j"'] & \arrow[d, phantom, "\Downarrow"] & X \\ & * \arrow[ur, "\alpha_j"'] & \end{tikzcd}$$

for the inclusions i_j with $[\alpha_j] = [\alpha]$ for all $1 \leqslant j \leqslant n$. Given a connected polyhedron X it is proved that $p\text{-}cat(X) = p\text{-}cat_{[\alpha]}(X)$ for some ray α (Proposition 1.6 in [8]) and that

$$\left| p\text{-}cat_{[\gamma]}(X) - p\text{-}cat_{[\gamma']}(X) \right| \leqslant 1$$

for any two strong ends $[\gamma]$, $[\gamma']$ (Proposition 3.4 in [9]).

In this paper we will also consider a weaker invariant of L-S type defined as follows. The *proper L-S category at infinity* of X is the least integer $p\text{-}cat^{\infty}(X) = n$ such that there are n inessential subpolyhedra $U_1, \ldots, U_n \subset X$ with $\bigcup_{i=1}^{n} U_i$ an ∞-neighbourhood of X. This number is not just a proper homotopy invariant but a homotopy invariant in the category $\mathcal{P}_\infty$ of germs at infinity of proper maps. The objects in this category are the spaces in $\mathcal{P}$ and morphisms (called *germs at* ∞) $X \nrightarrow Y$ are equivalence classes of proper maps defined from ∞-neighbourhoods of X into Y, two proper maps being equivalent if they coincide on some ∞-neighbourhood. A *homotopy at* ∞ is a homotopy $F : f \nRightarrow g$

$$\begin{tikzcd} X \arrow[r, bend left, "f" description, "/"] \arrow[r, bend right, "g" description] \arrow[r, phantom, "F \Downarrow"] & Y \end{tikzcd}$$

in $\mathcal{P}_\infty$; i.e., a germ $F : I \times X \nrightarrow Y$ with $F(0, -) = f$ and $F(1, -) = g$. We use the notation $[X, Y]_\infty$ for the set of homotopy classes of germs and $[X, Y]_\infty^{\mathbb{R}_+}$ for homotopy classes relative to the base ray. Notice that the identification $[\mathbb{R}_+, X]_\infty = [\mathbb{R}_+, Y]$ allows us to consider based spaces in $\mathcal{P}_\infty$ as based spaces in $\mathcal{P}$. One can also define the *proper L-S category at* ∞ *based at a strong end* $[\alpha]$. We do not use it in this paper.

Remark 2.1. It is obvious that $p\text{-}cat^{\infty}(X) \leqslant p\text{-}cat(X)$ and similarly for the based case. Moreover, if X is contractible the equality holds. Indeed, if $p\text{-}cat^{\infty}(X) \leqslant n$ there exists a subpolyhedron $U \subset X$ which is a ∞-neighbourhood in X and such that $U = \bigcup_{j=1}^{n} U_i$ with each inclusion

$$\begin{tikzcd} U_j \arrow[rr, "i_j"] \arrow[dr, "r_j"'] & \arrow[d, phantom, "H_j \Downarrow"] & X \\ & \mathbb{R}_+ \arrow[ur, "\alpha_j"'] & \end{tikzcd}$$

inessential in X. Moreover we can assume that r_j is a restriction of certain map $r : X \to \mathbb{R}_+$ for all j and $r(\overline{X - U}) = 0 \in \mathbb{R}_+$. For this we use the PHEP. Then the contractibility of

X allows us to extend the union of the homotopy H_1 and the inclusion $k_1 : \widetilde{U}_1 = U_1 \cup \overline{X - U} \subset X$ to a homotopy

$$\begin{tikzcd}
\widetilde{U}_1 \arrow[rr, "k_1"] \arrow[dr, "r"'] & \arrow[d, phantom, "\Downarrow"] & X \\
& \mathbb{R}_+ \arrow[ur, "\alpha_1"'] &
\end{tikzcd}$$

which shows that $\widetilde{U}_1$ is inessential in X, and so $p\text{-}cat(X) \leqslant n$.

3. Proper algebraic topology

Following [11] we use towers of groups to define the algebraic invariants in proper homotopy theory which are the analogues of the homotopy groups, and we state some basic results about "properly aspherical spaces" needed in the proofs of the main results of the paper.

Recall that given a category $\mathcal{C}$, the *category of towers* of $\mathcal{C}$, *tow*-$\mathcal{C}$, is the category of inverse sequences $\underline{X} = \{X_1 \leftarrow X_2 \leftarrow \cdots\}$ in $\mathcal{C}$ and pro-morphisms. See [20] for details. We will also use the full subcategory of $\mathcal{M}or(\textit{tow-}\mathcal{C})$ whose objects are arrows $\underline{f} : \underline{X} \to A$ where $\underline{X}$ is a (*tow*-$\mathcal{C}$)-object and A is a $\mathcal{C}$-object regarded as a constant tower whose bonding maps are the identity. This category is denoted $(\mathcal{C}, \textit{tow-}\mathcal{C})$. The object $\underline{f} : \underline{X} \to A$ can be represented as a tower $\{A \leftarrow X_{n_1} \leftarrow X_{n_2} \leftarrow \cdots\}$ for some subsequence $n_1 < n_2 < \cdots$ with A as a fixed object, and a morphism from $\underline{f} : \underline{X} \to A$ to $g : \underline{Y} \to B$ can be regarded as a $\mathcal{C}$-morphism between A and B together with a (*tow*-$\mathcal{C}$)-morphism from $\underline{X}$ to $\underline{Y}$ such that both morphisms are compatible via the bonding maps. Morphisms in $(\mathcal{C}, \textit{tow-}\mathcal{C})$ will be also called *pro-morphisms*.

Let $\mathcal{C} = \mathcal{G}r$ be the category of groups, given a properly based space (X, α) in $\mathcal{P}$, the *nth homotopy pro-group* of (X, α) is the object in $(\mathcal{G}r, \textit{tow-}\mathcal{G}r)$

$$\textit{pro-}\pi_n(X, \alpha) = \big\{\pi_n(X, x_0) \leftarrow \pi_n(U_1, x_1) \leftarrow \pi_n(U_2, x_2) \leftarrow \cdots\big\}$$

where $\{U_j\}$ is a system of ∞-neighbourhoods, $x_j = \alpha(t_j)$ with $\alpha([t_j, \infty)) \subset U_j$, and the bonding morphisms are induced by inclusions and base-point change isomorphisms along the ray α. For $n = 1$, $\textit{pro-}\pi_1(X, \alpha)$ is called the *fundamental pro-group* of (X, α). It is well known that the isomorphism type of $\textit{pro-}\pi_n(X, \alpha)$ only depends on the strong end $[\alpha]$. Similarly we can consider the same tower $\textit{pro-}\pi_n(X, \alpha)$ in the category *tow*-$\mathcal{G}r$ to obtain an "invariant at infinity" of X; i.e., a homotopy invariant in $\mathcal{P}_\infty$.

There are also proper homotopy invariants in $\mathcal{P}$ and $\mathcal{P}_\infty$ which are ordinary abelian groups. Examples of the latter are homology and cohomology of ends (denoted H_*^e and H_e^* respectively) while homology of infinite cycles (H_*^∞) and cohomology of compact supports (H_c^*) are examples of the former; see [16] or [21].

In the proof of Theorem A it is crucial that Whitehead manifolds are properly aspherical. More precisely,

Definition 3.1. A connected locally compact one-ended polyhedron X is said to be *properly aspherical at infinity* if, for some ray $\alpha : \mathbb{R}_+ \to X$, $\textit{pro-}\pi_n(X, \alpha) = 0$ is trivial in *tow*-$\mathcal{G}r$ for all $n \geqslant 2$. In addition X is said to be *properly aspherical* if it is aspherical and properly aspherical at infinity; that is, if $pro\text{-}\pi_n(X,\alpha)=0$ is trivial in $(\mathcal{G}r, tow\text{-}\mathcal{G}r)$ for all $n \geqslant 2$.

The role of the base ray is irrelevant in the previous definition as the following proposition shows.

Proposition 3.2. *If $pro\text{-}\pi_n(X,\alpha) = 0$ in $tow\text{-}\mathcal{G}r$ for some ray $\alpha:\mathbb{R}_+ \to X$ then $pro\text{-}\pi_n(X,\beta)=0$ for any other ray $\beta:\mathbb{R}_+ \to X$.*

Proof. Let $\{W_j\}$ be a system of ∞-neighbourhoods of X, α as in Definition 3.1, and β any ray. By using the PHEP, we can assume without loss of generality that $x_j=\alpha(t_j)=\beta(t_j)$ and $\alpha([t_j,\infty)) \cup \beta([t_j,\infty)) \subset W_j$.

It is well known (see [20]) that a tower of groups $\underline{G}=\{G_0 \leftarrow G_1 \leftarrow \cdots\}$ is trivial in $tow\text{-}\mathcal{G}r$ if and only if for any j there exists $n(j)>j$ such that the bonding homomorphism $G_{n(j)} \to G_j$ is trivial. In our case, the groups of the towers $pro\text{-}\pi_n(X,\alpha)$ and $pro\text{-}\pi_n(X,\beta)$ coincide and the bonding homomorphisms fit in diagrams

$$\begin{tikzcd}
 & \pi_n(U_k,x_k) \arrow[dd, "\simeq"', "\varphi"] \\
\pi_{n+1}(U_{k+1},x_{k+1}) \arrow[ur, "\text{bond } \alpha"] \arrow[dr, "\text{bond } \beta"'] & \\
 & \pi_n(U_k,x_k)
\end{tikzcd}$$

where φ is the automorphism induced by the loop defined by going from x_k to x_{k+1} by α and then going back by β. The previous criterion shows that $pro\text{-}\pi_n(X,\beta)=0$ is trivial. $\square$

Examples of properly aspherical spaces are the 1-dimensional spherical objects. More precisely, if E is a discrete set and $\alpha: E \to \mathbb{N} \subset \mathbb{R}_+$ is a proper map, the *n-dimensional spherical object S^n_α* is obtained by pasting a copy of the (based) n-sphere at $\alpha(e)$ for each $e \in E$. Notice that S^n_α is canonically well-based by the inclusion $\mathbb{R}_+ \subset S^n_\alpha$. The fundamental pro-group of S^1_α is isomorphic to the tower of free groups

$$\underline{L}_\alpha = \{\star_{\alpha(e)\geqslant 0}\mathbb{Z}e \hookleftarrow \star_{\alpha(e)\geqslant 1}\mathbb{Z}e \hookleftarrow \cdots\}$$

where the bonding homomorphisms are the obvious basis inclusions. These towers are termed *free towers*. The Euclidean plane is another example of properly aspherical space. Its fundamental pro-group belongs to a large class of towers of free groups termed telescopic towers. Namely, a *telescopic tower*

$$\underline{P} = \{P_0 \twoheadleftarrow P_1 \twoheadleftarrow P_2 \twoheadleftarrow \cdots\}$$

consists of free groups P_i of countable basis D_i such that $D_{i-1} \subset D_i$ with finite (possibly empty) differences $D_i - D_{i-1}$, and whose bonding homomorphisms are the obvious projections. For every telescopic tower $\underline{P}$ we can easily construct a properly aspherical space with fundamental pro-group isomorphic to $\underline{P}$ by attaching along the half-line $\mathbb{R}_+$

copies of infinite cylinders $S^1 \times [0, \infty)$ and planes $\mathbb{R}^2$ in a convenient locally finite way; compare [9].

In general the proper wedge of two properly aspherical (at ∞) spaces $X \vee_{\mathbb{R}_+} Y$ is a new properly aspherical (at ∞) space whose fundamental pro-group is the coproduct of *pro*-$\pi_1(X, \alpha)$ and *pro*-$\pi_1(Y, \beta)$ in $(\mathcal{G}r, \textit{tow-}\mathcal{G}r)$ ($\textit{tow-}\mathcal{G}r$, respectively). This coproduct, *pro*-$\pi_1(X, \alpha) \vee$ *pro*-$\pi_1(Y, \beta)$, is given by the levelwise free product of groups. In particular we have 2-dimensional properly aspherical spaces with fundamental pro-groups $\underline{L} \vee \underline{P}$ for every free tower $\underline{L}$ and telescopic tower $\underline{P}$.

The following proposition characterizes the set of homotopy classes of proper maps of one-ended spaces into aspherical spaces in terms of fundamental pro-groups.

Proposition 3.3. *Let (X, α) be a properly based connected locally compact one-ended polyhedron. If (Y, β) is a properly based aspherical space, then the fundamental pro-group functor induces a natural bijection*

$$[X, Y]^{\mathbb{R}_+} \cong \mathrm{Hom}\big(\textit{pro-}\pi_1(X, \alpha), \textit{pro-}\pi_1(Y, \beta)\big)$$

where "Hom" *stands for the morphism set in $(\mathcal{G}r, \textit{tow-}\mathcal{G}r)$.*

Remark 3.4. In the proof of Proposition 3.3 we use that any properly based connected locally compact one-ended polyhedron (Y, β) can be *reduced and normalized*; that is, there exists a (based) proper homotopy equivalence $(Y, \beta) \simeq (X, \alpha)$ where X is a CW-complex X whose 1-skeleton $X^1 = S^1_\gamma$ is a spherical object, and the $(n+1)$-skeleton X^{n+1} is the proper cofiber (i.e., the mapping cone, see Section 5 below) of a proper based map $S^n_\delta \to X^n$. The base ray of X is $\mathbb{R}_+ \subset S^1_\gamma = X^1 \subset X$, so X is well based; see IV.5.1 in [5] for details.

Proof of Proposition 3.3. Assume that X is already reduced and normalized and α is its canonical ray. If X is one-dimensional then it is a spherical object $X = S^1_\gamma$ and the result is well known; see V.3.10 in [5]. For $\dim X \geqslant 2$ the result follows by applying inductively on skeletons the cofiber sequence of the attaching maps of the cells in X. This sequence is available in any cofibration category; see I.7.6 in [5] for details. □

Since for any reduced and normalized CW-complex we can choose a system of ∞-neighbourhoods consisting of reduced and normalized subcomplexes, one can check in a similar way to Proposition 3.3 the following

Proposition 3.5. *Let (X, α) be a properly based connected locally compact one-ended polyhedron. If (Y, β) is a properly based aspherical at ∞ space, then the fundamental pro-group functor induces a natural bijection*

$$[X, Y]^{\mathbb{R}_+}_\infty \cong \mathrm{Hom}_\infty\big(\textit{pro-}\pi_1(X, \alpha), \textit{pro-}\pi_1(Y, \beta)\big)$$

where "Hom_∞" *stands for the morphism set in $\textit{tow-}\mathcal{G}r$.*

Remark 3.6. According to Proposition 3.3 the (based) proper homotopy type of a properly aspherical space (X, α) is determined by the isomorphism class of its fundamental pro-group $\underline{G} = \textit{pro-}\pi_1(X, \alpha)$ in $(\mathcal{G}r, \textit{tow-}\mathcal{G}r)$, therefore we simply denote $B\underline{G} = X$. Although

$B\underline{G}$ does not exist for an arbitrary tower $\underline{G}$, we have already seen that, for any free tower $\underline{L}$ and any telescopic tower $\underline{P}$, there exists a 2-dimensional model $B(\underline{L} \vee \underline{P})$.

4. Whitehead manifolds

Here we restate and prove Theorem A in the Introduction. For this we collect some basic homotopical properties of the Whitehead manifolds. We start with the following proposition; compare the proof of Theorem 3.4 in [6].

Proposition 4.1. *Any Whitehead* 3*-manifold $W \neq \mathbb{R}^3$ satisfies that pro-$\pi_2(X,\alpha) = 0$ for any ray α.*

Proof. As it was shown by McMillan [23] any Whitehead 3-manifold W is an increasing union, $W = \bigcup_{n\geqslant 1} H_n$ of handlebodies (only 0- and 1-handles). In case $\pi_2(W - \operatorname{int} H_{n_k}) \neq 0$ for a sequence $n_1 \leqslant n_2 \leqslant \ldots$, the sphere theorem (4.A.4 in [25]) provides us with an embedded sphere $S_k^2 \subset W - \operatorname{int} H_{n_k}$ which is not nullhomotopic. However, since W is irreducible there exists a ball $B_k^3 \subset W$ with $\partial B_k^3 = S_k^2$. Hence $H_{n_k} \subset B_k^3$ and $W = \bigcup_{k\geqslant 1} B_k^3 \cong \mathbb{R}^3$ is homeomorphic to $\mathbb{R}^3$ by the collar theorem (2.F.10 in [25]). $\square$

Proposition 4.2. *Let M^3 be any connected one-ended open* 3*-manifold with pro-$\pi_2(M,\alpha) = 0$ in tow-$\mathcal{G}r$ for some ray α. Then M is properly aspherical at ∞.*

Proof. Let $\{U_i\}$ be a system of ∞-neighbourhoods of M consisting of connected submanifolds and such that the bonding homomorphisms

$$\pi_2\big(U_i, \alpha(t_i)\big) \xrightarrow{k_{i*}} \pi_2\big(U_{i-1}, \alpha(t_i)\big) \xrightarrow{\alpha_i^{\#}} \pi_2\big(U_{i-1}, \alpha(t_{i-1})\big)$$

are trivial. Here $\alpha_i^{\#}$ is the base point change isomorphism induced by $\alpha_i = \alpha|[t_{i-1}, t_i]$. Let $\widetilde{U}_i$ be the universal covering space of U_i. According Lemma 4.3 below, the manifold $\widetilde{U}_i$ has the same homotopy type as a wedge of 2-spheres. Hence the triviality of the bonding homomorphisms yields that the induced maps $\tilde{k}_i : \widetilde{U}_i \to \widetilde{U}_{i-1}$ are homotopically trivial and hence *pro-*$\pi_q(M,\alpha) = 0$ in *tow-*$\mathcal{G}r$ for all $q \geqslant 2$. $\square$

Lemma 4.3. *Any simply connected non-compact* 3*-manifold with boundary N has the same homotopy type as a wedge of* 2*-spheres.*

Proof. There is an exact sequence

$$0 = H_3(N, \partial N) \to H_2(\partial N) \to H_2(N) \to H_2(N, \partial N). \tag{2}$$

Moreover, by the Lefschetz duality $H_2(N, \partial N) \cong H_c^1(N)$ where the first cohomology group with compact supports is the direct limit $H_c^1(N) = \varinjlim H^1(N, U_j)$ for a system of ∞-neighbourhoods $\{U_j\}_{j\geqslant 0}$ with $U_0 = N$. In fact, since N is simply connected $H^1(N, U_j) = \operatorname{Coker}[k_j^* : H^0(N) \to H^0(U_j)]$ and we have a short exact sequence

$$0 \to H^0(N) \xrightarrow{k} \varinjlim H^0(U_j) \to H_c^1(N) \to 0$$

where k is the homomorphism induced by the k_j's and $H_e^0(N) = \varinjlim H^0(U_j)$ is the 0th cohomology group of ends of N which is free; see 3.7.2 and 3.9.12 in [16]. Moreover the sequence splits and hence $H_c^1(N) \cong H_2(N, \partial N)$ is also free. To show the splitting, we fix a Freudenthal end $\varepsilon_0 = \{N = C_0 \supset C_1 \supset \cdots\}$ with $C_j \subset U_j$ a component identified to a generator of $H^0(U_j)$. Then we define a retraction $\rho : H_e^0(N) \to H^0(N)$ of k as the homomorphism induced by the projections $H^0(U_j) \to H^0(N)$ which carry C_j to N and the rest of components to 0.

As $H_2(\partial N)$ is also free since N is orientable, the sequence (2) above yields that $H_2(N)$ is free. Moreover, if $f : \bigvee_A S^2 \to N$ is a map inducing an isomorphism in H_s for all s, the homological Whitehead theorem shows that f is a homotopy equivalence. $\square$

Next we state the following theorem, from which we derive Theorem A by using Proposition 4.5. For this we recall that a tower of groups $\underline{G} = \{G_0 \leftarrow G_1 \leftarrow \cdots\}$ is said to be *essentially monomorphic* if it is isomorphic in *tow-$\mathcal{G}r$* to a tower whose bonding homomorphisms are injective.

Theorem 4.4. *Let W^3 be a connected one-ended open* 3*-manifold with pro-$\pi_2(W, \alpha) = 0$ in tow-$\mathcal{G}r$ for some ray α. If pro-$\pi_1(W, \beta)$ is essentially monomorphic and*

$$\varprojlim \textit{pro-}\pi_1(X, \beta) = 0$$

is trivial for any ray β then p-$cat^\infty(W) = 4$, and hence p-$cat(W)$ is also 4.

Proof. By Proposition 4.2 W is properly aspherical at ∞. Assume $p\text{-}cat^\infty(W) \leqslant 3$. By Theorem 5.6 in Section 5 we have a diagram in $\mathcal{P}_\infty$

$$\begin{tikzcd} W \arrow[rr, equal] \arrow[dr] & & W \\ & W^2 \arrow[ur, tail] & \end{tikzcd}$$

where W^2 is the 2-skeleton of W. This diagram yields a commutative diagram in homology of ends

$$\begin{tikzcd} \mathbb{Z} \cong H_3^e(W) \arrow[rr, equal] \arrow[dr] & & H_3^e(W) \\ & 0 = H_3^e(W^2) \arrow[ur] & \end{tikzcd}$$

which reaches a contradiction. Here $H_3^e(W^2) = 0$ by dimension reasons and $H_3^e(W) \cong H_3^\infty(W) \cong H^0(W) \cong \mathbb{Z}$ by the contractibility of W, the long exact sequence $H_* \to H_*^\infty \to H_*^e$ (3.5.1 in [16] or VI.4.5 in [5]) and the Poincaré duality (6.7.4 in [16] or 11.4 in [21]). Finally the inequalities $p\text{-}cat^\infty(W) \leqslant p\text{-}cat(W) \leqslant \dim W + 1$ show that $p\text{-}cat(W)$ is 4 as well. $\square$

For a space X in $\mathcal{P}$ it is immediate to check that *pro-$\pi_1(X, \beta)$* is essentially monomorphic if and only if X is eventually π_1-injective at ∞ as defined in [27]. Moreover, any open

3-manifold M is eventually π_1-injective at ∞ if and only if it is *eventually end-irreducible*; that is, there exists an exhausting sequence $\{M_n\}$ in M consisting of compact connected submanifolds and such that there exists n_0 for which each surface ∂M_n is incompressible in $\overline{M - M_{n_0}}$ if $n > n_0$; see 2.1 in [6].

For Whitehead manifolds the vanishing of the inverse limits in Theorem 4.4 is equivalent to the geometrical criterion of $\mathbb{R}^2$-irreducibility. Recall that an open manifold M is said to be $\mathbb{R}^2$-*irreducible* if any properly embedded plane $\mathbb{R}^2 \subset M$ bounds a half-space $\mathbb{R}^2 \times \mathbb{R}_+ \subset M$. We have

Proposition 4.5. *Let M^3 be a Whitehead manifold. Then M is $\mathbb{R}^2$-irreducible if and only if $\varprojlim$ pro-$\pi_1(M, \alpha) = 0$ for any ray $\alpha : \mathbb{R}_+ \to M$.*

Proof. As $\pi_1(M) = 0$ and *pro*-$\pi_1(\mathbb{R}^2, \mathbb{R}_+) = \{0 \leftarrow \mathbb{Z} = \mathbb{Z} = \cdots\}$ we have

$$\left[\mathbb{R}^2, M\right]^{\mathbb{R}_+} = \varprojlim \textit{pro-}\pi_1(M, \alpha)$$

by Proposition 3.3 and 4.6.10 in [11]. Therefore, if $\varprojlim$ *pro*-$\pi_1(M, \alpha) \neq 0$ is not trivial, the plane theorem (2.2 in [7]) yields a non-trivial proper plane $\mathbb{R}^2 \subset M$ and M is not $\mathbb{R}^2$-irreducible.

Conversely, if $\mathbb{R}^2 \subset M$ is a non-trivial plane in M, by 4.1 in [26] we know that this plane is not properly homotopically trivial in M, and hence it represents a non-trivial element in $\varprojlim$ *pro*-$\pi_1(M, \beta)$ for $\beta : \mathbb{R}_+ \subset \mathbb{R}^2 \subset M$. □

Next remark provides us with sufficiently many examples of open 3-manifolds for which Theorem 4.4, and hence Theorem A, holds.

Remark 4.6. There exist uncountably many eventually end irreducible, $\mathbb{R}^2$-irreducible Whitehead manifolds with different proper homotopy types. Indeed, McMillan [22] defines an uncountable family of Whitehead manifolds which are the increasing union of solid tori. Moreover they are of finite genus at infinity, and hence eventually end-irreducible (2.3 in [6]). Therefore they have pairwise different proper homotopy types by 3.4 in [6]. Now one uses 4.2 in [26] to convince oneself that not only the classical Whitehead 3-manifold but all Whitehead manifolds constructed by McMillan are in fact $\mathbb{R}^2$-irreducible; see Fig. 1 in [22].

5. A proper Eilenberg–Ganea theorem for a class of aspherical polyhedra

In this section we prove Theorem 5.6 which as was pointed out in the Introduction is a partial analogue in $\mathcal{P}_\infty$ of the Eilenberg–Ganea theorem stating that the L-S category of an aspherical space X is $\leqslant n$ if and only if the identity 1_X factorizes through an $(n-1)$-polyhedron.

Because of the lack of some products and fibrations in proper homotopy theory, the proper L-S category (at ∞) is not characterized by fat wedges (Whitehead [28]) or fibrations (Ganea [15]). However the computations in Theorem 5.6 are carried out by using the universal property in Proposition 5.2 which allows to replace the homotopy fiber in the Ganea fibrations. For this, we recall some important features of the homotopy theory of the proper categories $\mathcal{P}$ and $\mathcal{P}_\infty$. As pointed out in [1], both categories fit into the axiomatic framework provided by the I-categories of Baues [4] which are examples of cofibration categories. This allows us to carry out the basic homotopical constructions in $\mathcal{P}$ and $\mathcal{P}_\infty$ as well. The I-category structures are given by the ordinary cylinder functor $IX = [0,1] \times X$ with inclusions $i_k : X \to IX$ $(k = 0, 1)$, $i_k(x) = (k, x)$, and projection $p : IX \to X$, $p(t, x) = x$. In this setting the *based cylinder*, the *cone*, and the *mapping cone* (*cofiber*) of a proper based map $f : X \to Y$ between well-based spaces in $\mathcal{P}$ are given by the push-out diagrams

$$
\begin{tikzcd}
I\mathbb{R}_+ \arrow[r, tail, "I\alpha"] \arrow[d, "p"'] \arrow[dr, phantom, "\text{push}"] & IX \arrow[d] \\
\mathbb{R}_+ \arrow[r, tail] & I_{\mathbb{R}_+}X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, tail, "\tilde{\imath}_1"] \arrow[d, "r"'] \arrow[dr, phantom, "\text{push}"] & I_{\mathbb{R}_+}X \arrow[d] \\
\mathbb{R}_+ \arrow[r, tail] & C_{\mathbb{R}_+}X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, tail, "\tilde{\imath}_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{push}"] & C_{\mathbb{R}_+}X \arrow[d] \\
Y \arrow[r, tail] & C_{\mathbb{R}_+}f
\end{tikzcd}
$$

respectively. Here $r : X \to \mathbb{R}_+$ is any proper map and $\tilde{\imath}_0$ and $\tilde{\imath}_1$ are the cofibrations induced by the inclusions (cofibrations) $i_0, i_1 : X \rightarrowtail IX$. Since all maps r are properly homotopic the (based) proper homotopy types of the push-outs above are independent of r. Moreover the *based suspension* of X, $\Sigma_{\mathbb{R}_+}X$, is obtained as the mapping cone of any r.

As usual, the addition of two homotopies $F : f \Rightarrow g$ and $G : g \Rightarrow h$ is the homotopy $F + G : f \Rightarrow h$ given by $(F + G)(t, -) = F(2t, -)$ for $t \leqslant 1/2$ and $(F + G)(t, -) = G(2t - 1, -)$ for $t \geqslant 1/2$. The inverse homotopy $-F : g \Rightarrow f$ of F is given by $(-F)(t, -) = F(1 - t, -)$. Finally the trivial homotopy $0_f : f \Rightarrow f$ is the map $0_f = fp : IX \to X \to Y$. Given a homotopy F between the maps $f, g : X \to Y$ and two maps $h : Y \to Z$ and $k : Z \to X$ the *pushforward* and *pullback* of F through h and k are the homotopies $h_*F = hF$ and $k^*F = F(Ik)$ respectively.

A *track* from f to g is an element of the set $[IX, Y]^{(f,g)}$ of homotopy classes of homotopies $f \Rightarrow g$ relative to the boundary $i_0X \cup i_1X \subset IX$ of the cylinder. The operations above between maps and homotopies induce well-defined operations between tracks and homotopy classes, which give rise to a groupoid enrichment of the proper category $\mathcal{P}$; compare I.5.3 in [5]. In particular, given a map $f : X \to Y$ the set $[IX, Y]^{(f,f)}$ is a group for the track addition.

The obvious forgetful functor $\mathcal{P} \to \mathcal{P}_\infty$ preserves push-outs and cylinders, hence it is a model functor in the sense of Baues (I.1.10 in [4]), so it is compatible with all the additional structure of a cofibration category obtained from the axioms.

In order to ease the reading we will denote the germs in the image of the forgetful functor (i.e., germs $X \nrightarrow Y$ defined by real proper maps from X to Y) by straight arrows as usual.

The base cylinder, cone and suspension of a space in $\mathcal{P}_\infty$ coincide with the corresponding construction in $\mathcal{P}$. Similarly the *cofiber* or *mapping cone* of a germ $f : X \nrightarrow Y$ is the following push-out in $\mathcal{P}_\infty$

$$
\begin{tikzcd}
X \arrow[r, tail, "\tilde{\imath}_0"] \arrow[d, "f"', "\backslash" marking] \arrow[dr, phantom, "\text{push}"] & C_{\mathbb{R}_+}X \arrow[d, "\backslash" marking] \\
Y \arrow[r, tail] & C_{\mathbb{R}_+}f
\end{tikzcd}
$$

Moreover, the operations above between homotopies and proper maps in $\mathcal{P}$ are also available in $\mathcal{P}_\infty$ in the obvious way, yielding a groupoid enrichment of $\mathcal{P}_\infty$ compatible with the forgetful functor. Given two germs $f, g : X \nrightarrow Y$ we write $[IX, Y]_\infty^{(f,g)}$ for the corresponding set of tracks in $\mathcal{P}_\infty$. The set $[IX, Y]_\infty^{(f,f)}$ is also a group for the track addition.

Remark 5.1. If f is a trivial map $f : X \xrightarrow{r} \mathbb{R}_+ \xrightarrow{\alpha} Y$ then the push-out diagram

$$\begin{tikzcd}
X \coprod X \arrow[r, tail, "(i_0,i_1)"] \arrow[d, "(r,r)"'] \arrow[dr, phantom, "\text{push}"] & IX \arrow[d] \\
\mathbb{R}_+ \arrow[r, tail, "\beta"'] & \Sigma_* X
\end{tikzcd}$$

yields an identification $[IX, Y]_\infty^{(f,f)} = [\Sigma_* X; Y]_\infty^{\mathbb{R}_+}$ where Y is properly based by α above. Notice that the *punctured torus* $\Sigma_* X$ is one-ended and $p\text{-}cat_{[\beta]}\,\Sigma_* X \leqslant 2$.

In the next proposition we show a "universal property" which plays the role of the homotopy fiber of the inclusion of the 1-skeleton of a properly aspherical space at ∞.

Proposition 5.2. *Let X be a reduced and normalized* CW*-complex with attaching map of* 2*-cells $f : S^1_\gamma \to S^1_\beta = X^1$. Assume*:

(a) *X is properly aspherical at ∞,*
(b) *pro-$\pi_1(X, \mathbb{R}_+)$ is essentially monomorphic, and*
(c) *$\varprojlim$ pro-$\pi_1(X, \mathbb{R}_+) = 0$.*

If a germ at ∞ $k : Y \nrightarrow S^1_\beta$ is inessential in X, factoring up to homotopy in $\mathcal{P}_\infty$ through the canonical ray $\delta : \mathbb{R}_+ \subset S^1_\beta = X^1 \subset X$, then there exists a diagram of homotopies in $\mathcal{P}_\infty$

$$\begin{tikzcd}
S^1_\beta \arrow[rr, tail] & & C_{\mathbb{R}_+} S^1_\gamma \vee S^1_\beta & & \\
\mathbb{R}_+ \arrow[u, tail] & & S^1_\gamma \vee S^1_\beta \arrow[u, tail] \arrow[ll, phantom, "\Leftarrow"] \arrow[dr, "{(f,1)}"] & & \\
& & Y \arrow[ull] \arrow[u] \arrow[r, "k"'] & S^1_\beta \arrow[ul, phantom, "\Uparrow"] \arrow[r, tail] & X
\end{tikzcd}$$

In the proof we will use the following lemmas

Lemma 5.3. *In the same conditions as in the statement of Proposition* 5.2*, there exists a connected one-ended based polyhedron $(\overline{Y}, \alpha)$ containing Y and an extension $\bar{k} : \overline{Y} \nrightarrow S^1_\beta$ of k such that $\bar{k}\alpha$ represents the canonical strong end of S^1_β and the composite $\overline{Y} \overset{\bar{k}}{\nrightarrow} S^1_\beta \rightarrowtail X$ is inessential.*

Proof. There exists a nullhomotopy

$$\begin{tikzcd} Y \arrow[r, "k", "/" marking] \arrow[d, "r"'] & S^1_\beta \arrow[d, tail] \arrow[dl, phantom, "\swarrow H"] \\ \mathbb{R}_+ \arrow[r, "\delta"'] & X \end{tikzcd} \qquad (*)$$

We call Y' to the one-point union of Y and $\mathbb{R}_+$, and $\alpha' : \mathbb{R}_+ \rightarrowtail Y'$ to the inclusion. One can attach segments at the vertices of Y' in a locally finite manner to join the components of the ∞-neighbourhoods of Y', obtaining in this way a connected one-ended space $\overline{Y}$. We call α to the composite

$$\mathbb{R}_+ \overset{\alpha'}{\rightarrowtail} Y' \subset \overline{Y}.$$

Since $[Z, \mathbb{R}_+]$ is always a singleton for every space Z in $\mathcal{P}$, there is an extension $\bar{r} : \overline{Y} \to \mathbb{R}_+$ of r. Moreover, by the PHEP and diagram $(*)$ there exists an inessential germ $k' : \overline{Y} \nrightarrow X$ extending $Y \overset{k}{\nrightarrow} S^1_\beta \rightarrowtail X$. As $\dim(\overline{Y} - Y) \leqslant 1$, the proper cellular approximation theorem (see [13] or IV.3.18 in [5]) ensures the existence of a germ $\bar{k} : \overline{Y} \nrightarrow S^1_\beta$ extending k and such that the composite

$$\overline{Y} \overset{\bar{k}}{\nrightarrow} S^1_\beta \rightarrowtail X$$

is homotopic to k'. Therefore the germ $\bar{k}$ satisfies the conditions of the statement. $\square$

We will also need the following purely algebraic lemma concerning essentially monomorphic towers.

Lemma 5.4. *If $\underline{G}$ is an essentially monomorphic tower of groups with $\varprojlim \underline{G} = 0$ and $\phi : \underline{L} \twoheadrightarrow \underline{G}$ is an epimorphism in tow-$\mathcal{G}r$ from a free tower $\underline{L}$ then the kernel of ϕ is a projective object.*

The proof of the lemma will be given in Appendix A.

Proof of Proposition 5.2. By Lemma 5.3 we can assume that Y is connected, one-ended and based by $\alpha : \mathbb{R}_+ \to Y$ in such a way that $k\alpha$ is the canonical strong end. Moreover the 2-skeleton of X can be described by the push-out diagram

$$\begin{tikzcd} S^1_\gamma \vee S^1_\beta \arrow[r, "{(f,1)}"] \arrow[d, tail] \arrow[dr, phantom, "\text{push}"] & S^1_\beta = X^1 \arrow[d, tail] \\ C_{\mathbb{R}_+} S^1_\gamma \vee S^1_\beta \arrow[r] & X^2 \end{tikzcd}$$

which restricts to diagrams defining a system of ∞-neighbourhoods of X^2 consisting of subcomplexes. By applying the Van Kampen theorem to these subcomplexes we obtain a commutative diagram in *tow-$\mathcal{G}r$*; compare V.1.16 in [5],

$$
\begin{tikzcd}
\underline{L}_\beta & & \\
\underline{L}_\gamma \vee \underline{L}_\beta \arrow[u, "{(0,1)}"] \arrow[dr, "{(f,1)_*}"] & & \\
\mathrm{Ker}(0,1) \arrow[u, hook, dotted] \arrow[r, tail, dotted] & \underline{L}_\beta \arrow[r, two heads, "\phi"] & \textit{pro-}\pi_1(X, \mathbb{R}_+)
\end{tikzcd}
\tag{3}
$$

where the vertical and horizontal sequences are exact. Moreover, the subdiagram of solid arrows can be realized by a diagram of aspherical spaces at ∞ and germs of proper maps

$$
\begin{tikzcd}
C_{\mathbb{R}_+} S^1_\gamma \vee S^1_\beta & & \\
S^1_\gamma \vee S^1_\beta \arrow[u, tail] \arrow[dr, "{(f,1)}"] & & \\
 & S^1_\beta \arrow[r, tail] & X
\end{tikzcd}
\tag{4}
$$

in the sense that the functor $[Y, -]_\infty^{\mathbb{R}_+}$ applied to diagram (4) coincides, via Proposition 3.5, with the functor $\mathrm{Hom}_\infty(\textit{pro-}\pi_1(Y, \alpha), -)$ applied to the subdiagram of solid arrows in (3). In particular $k_*: \textit{pro-}\pi_1(Y, \alpha) \to \underline{L}_\beta$ factorizes through $\mathrm{Ker}\,\phi$. Moreover, Lemma 5.4 shows that $\mathrm{Ker}\,\phi$ is a projective object in *tow-$\mathcal{G}r$*, hence, since the row in (3) is exact, the morphism k_* factors in the following way in *tow-$\mathcal{G}r$*

$$
\begin{tikzcd}
\mathrm{Ker}(0,1) \arrow[r] & \underline{L}_\beta \\
 & \textit{pro-}\pi_1(Y, \alpha) \arrow[ul, "\psi"] \arrow[u, "k_*"']
\end{tikzcd}
$$

Finally, by Proposition 3.5, the composite

$$
\textit{pro-}\pi_1(Y, \alpha) \xrightarrow{\psi} \mathrm{Ker}(0,1) \hookrightarrow \underline{L}_\gamma \vee \underline{L}_\beta
$$

can be realized by a germ $Y \nrightarrow S^1_\gamma \vee S^1_\beta$ for which the proposition follows. $\square$

Proposition 5.5. *Let X be a reduced and normalized* CW*-complex under the assumptions of Proposition* 5.2*. For any diagram in* $\mathcal{P}_\infty$

$$\begin{tikzcd} Y \arrow[r, "k", "/"{marking}] \arrow[d, "r"'] \arrow[dr, phantom, "\not\Swarrow K"] & X^1 \arrow[d, tail] \\ \mathbb{R}_+ \arrow[r, "\delta"'] & X^1 \subset X \end{tikzcd}$$

there exists another one

$$\begin{tikzcd} Y \arrow[r, "k", "/"{marking}] \arrow[d, "r"'] \arrow[dr, phantom, "\not\Swarrow \widetilde{K}"] & X^1 \arrow[d, tail] \\ \mathbb{R}_+ \arrow[r, "\delta"'] & X^1 \subset X^2 \end{tikzcd}$$

such that if $n : X^2 \rightarrowtail X$ is the inclusion we have $n_* \widetilde{K} = K$ as tracks.

Proof. Since X^2 is given by the following push-out

$$\begin{tikzcd} S^1_\gamma \vee S^1_\beta \arrow[r, "(f,1)"] \arrow[d, tail] \arrow[dr, phantom, "\text{push}"] & S^1_\beta = X^1 \arrow[d, tail] \\ C_{\mathbb{R}_+} S^1_\gamma \vee S^1_\beta \arrow[r] & X^2 \end{tikzcd}$$

the diagram in Proposition 5.2 yields a homotopy

$$\begin{tikzcd} Y \arrow[r, "k", "/"{marking}] \arrow[d, "r"'] \arrow[dr, phantom, "\not\Swarrow"] & X^1 \arrow[d, tail] \\ \mathbb{R}_+ \arrow[r, "\delta"'] & X^2 \end{tikzcd}$$

Unfortunately the pushforward of this track by n needs not be represented by K. Nevertheless, the previous homotopy shows that the set $[IY, X^2]_\infty^{(\delta r, k)}$ is not empty, and in this case we know that the group $[IY, X^2]_\infty^{(\delta r, \delta r)}$ acts effectively and transitively on $[IY, X^2]_\infty^{(\delta r, k)}$ in a natural way; see I.5 in [5]. Moreover there is a natural identification $[IY, X^2]_\infty^{(\delta r, \delta r)} = [\Sigma_* Y, X^2]_\infty^{\mathbb{R}_+}$ (see Remark 5.1) and similarly by replacing X^2 by X. Moreover, the homomorphism

$$n_* : \big[\Sigma_* Y, X^2\big]_\infty^{\mathbb{R}_+} \twoheadrightarrow [\Sigma_* Y, X]_\infty^{\mathbb{R}_+} \tag{5}$$

is surjective. Indeed, since $p\text{-}cat_{\mathbb{R}_+}(\Sigma_* Y) \leqslant 2$ and X is aspherical at ∞, any based germ $\Sigma_* Y \nrightarrow X$ factorizes up to homotopy in $\mathcal{P}_\infty$ through $B(pro\text{-}\pi_1(\Sigma_* Y, \mathbb{R}_+))$; compare 6.4 in [9], Proposition 3.5 and Remark 3.6. As $B(pro\text{-}\pi_1(\Sigma_* Y, \mathbb{R}_+))$ has dimension 2, the proper cellular approximation theorem yields that n_* in (3) is a surjection. Therefore

$$n_* : \big[IY, X^2\big]_\infty^{(\delta r, k)} \twoheadrightarrow [IY, X]_\infty^{(\delta r, k)}$$

is also surjective and the result follows. □

We are now ready to prove

Theorem 5.6. *Let X be a polyhedron which is properly aspherical at ∞. Assume*:

(a) $p\text{-}cat^{\infty}(X) \leqslant 3$,
(b) *pro-*$\pi_1(X,\alpha)$ *is essentially monomorphic, and*
(c) $\varprojlim$ *pro-*$\pi_1(X,\alpha) = 0$ *for any ray* $\alpha : \mathbb{R}_+ \to X$.

Then there exists a diagram in $\mathcal{P}_\infty$

$$\begin{tikzcd} X \arrow[rr, equal] \arrow[dr, "\shortmid" marking] & \arrow[d, phantom, "\Downarrow"] & X \\ & X^2 \arrow[ur] & \end{tikzcd}$$

which gives a factorization up to homotopy of the identity 1_X *through the* 2*-skeleton* X^2.

For the proof of Theorem 5.6 we use the following

Lemma 5.7. *If* $\underline{G}$ *is a tower in tow-*$\mathcal{G}r$ *with* $\varprojlim \underline{G} = 0$ *and* $\underline{P}$ *is a telescopic tower then* $\mathrm{Hom}_\infty(\underline{P}, \underline{G}) = 0$.

Proof. Let $D = \bigcup_{n \geqslant 0} D_n$ be the union of all basis involved in the telescopic tower $\underline{P}$. We define the constant tower $\underline{P}'$ of free groups $P'_n = \star_{d \in D}\mathbb{Z}$ for each $n \geqslant 0$ and the identity $P'_{n+1} \to P'_n$ as bonding homomorphism. The obvious epimorphism $\underline{P}' \twoheadrightarrow \underline{P}$ yields an injection

$$\mathrm{Hom}_\infty(\underline{P}, \underline{G}) \hookrightarrow \mathrm{Hom}_\infty(\underline{P}', \underline{G}) = \prod_{d \in D} \mathrm{Hom}_\infty(\underline{\mathbb{Z}}, \underline{G})$$

for the constant tower $\{\mathbb{Z} = \mathbb{Z} = \cdots\}$. Now the result follows since $\mathrm{Hom}_\infty(\underline{\mathbb{Z}}, \underline{G}) = \varprojlim \underline{G} = 0$ by 4.6.10 in [11]. □

Proof of Theorem 5.6. As $p\text{-}cat^{\infty}(X) \leqslant 3$ there exists a subpolyhedron $\Omega \subset X$ which is an ∞-neighbourhood in X and such that $\Omega = U_1 \cup U_2 \cup U_3$ with each U_k inessential in X. Therefore, for $1 \leqslant k \leqslant 3$ we have homotopies

$$\begin{tikzcd} U_k \arrow[rr, "j_k"] \arrow[dr, "r_k"'] & \arrow[d, phantom, "H_k \Downarrow"] & X \\ & \mathbb{R}_+ \arrow[ur, "\alpha_k"'] & \end{tikzcd} \tag{6}$$

where j_k is the corresponding inclusion. By Remark 3.4 we can assume that X^2 is the cofiber of a proper map $f : S^1_\alpha \to S^1_\beta = X^1$ and α_3 in (6) is the inclusion $\mathbb{R}_+ \subset S^1_\beta \subset X$.

Moreover, the homotopies H_k in (6) induce proper maps $h_k : CU_k \to X$ from the unreduced proper cone of U_k into X which fit in the following commutative diagrams ($1 \leqslant k \leqslant 3$)

$$\begin{tikzcd}
U_k \arrow[r, "r_k"] \arrow[d, tail, "i_1"', "\simeq"] \arrow[dr, phantom, "\text{push}"] & \mathbb{R}_+ \arrow[d, tail, "\simeq"] \arrow[ddr, bend left, "\alpha_k"] & \\
IU_k \arrow[r] \arrow[drr, bend right, "H_k"'] & CU_k \arrow[dr, "h_k"] & \\
U_k \arrow[u, tail, "i_0"] \arrow[ur] & & X
\end{tikzcd} \tag{7}$$

The gluing of the (unreduced) cones CU_1 and CU_2 along $U_1 \cap U_2$ gives us the commutative diagram

$$\begin{tikzcd}
U_1 \cap U_2 \arrow[r, tail] \arrow[d, tail] & U_2 \arrow[r, tail] & CU_2 \arrow[dd] \arrow[dddr, bend left, "h_2"] & \\
U_1 \arrow[d, tail] & \text{push} & & \\
CU_1 \arrow[rr, tail] \arrow[drrr, bend right, "h_1"'] & & CU_1 \cup CU_2 \arrow[dr, "h_1 \cup h_2"'] & \\
& & & X
\end{tikzcd} \tag{8}$$

Moreover the restriction $U_1 \cup U_2 \subset CU_1 \cup CU_2 \xrightarrow{h_1 \cup h_2} X$ coincides with the inclusion $j_1 \cup j_2 : U_1 \cup U_2 \subset X$. The polyhedron $CU_1 \cup CU_2$ (which can be assume to be one-ended[1]) has proper L-S category $\leqslant 2$ since CU_k is properly homotopically equivalent to $\mathbb{R}_+$; see diagram (7). Therefore, by 1.6 in [8] and 6.4 in [9] there exists a base ray $\delta : \mathbb{R}_+ \to CU_1 \cup CU_2$ such that the fundamental pro-group $\textit{pro-}\pi_1(CU_1 \cup CU_2, \delta) = \underline{L} \vee \underline{P}$ is a coproduct of a free tower $\underline{L}$ and a telescopic tower $\underline{P}$. Then Proposition 3.3 provides us with a (based) proper map $m_1 : CU_1 \cup CU_2 \to B(\underline{L} \vee \underline{P})$ inducing the identity on fundamental pro-groups and $h_1 \cup h_2$ factorizes up to homotopy through m_1 and a proper map $m_2 : B(\underline{L} \vee \underline{P}) \to X$. By Lemma 5.7, Proposition 3.3 and the vanishing of the inverse limits in the statement of the theorem, the map m_2 factorizes up to homotopy in the following way

$$B(\underline{L} \vee \underline{P}) \xrightarrow{(1,0)} B\underline{L} \to X$$

1 As any locally finite family of inessential compact subpolyhedra can be deformed to any ray in a one-ended space, we can assume without loss of generality that the U_k's are non-compact. Moreover, if the intersection $U_1 \cap U_2$ is compact we can add to U_2 a locally finite sequence $A \subset U_1 - U_2$ to get a new inessential subpolyhedron $\widetilde{U}_2 = U_2 \cup A$ for which the intersection $U_1 \cap \widetilde{U}_2$ is non-compact, and hence $CU_1 \cup C\widetilde{U}_2$ is one-ended.

where $B\underline{L}$ is a 1-dimensional spherical object. Hence by the proper cellular approximation theorem $h_1 \cup h_2$ factorizes up to homotopy through the 1-skeleton X^1 and so the inclusion $U_1 \cup U_2 \subseteq X$ fits in a diagram

$$\begin{tikzcd}
U_1 \cup U_2 \arrow[r, "g"] \arrow[d, tail, "j_1 \cup j_2"'] \arrow[dr, phantom, "\overset{G}{\Rightarrow}"] & X^1 \arrow[dl, tail, bend left] \\
X &
\end{tikzcd}$$

Let $l_1 : (U_1 \cup U_2) \cap U_3 \rightarrowtail U_1 \cup U_2$ and $l_2 : (U_1 \cup U_2) \cap U_3 \rightarrowtail U_3$ be the inclusions. By applying Proposition 5.5 to the map $k = gl_1$ and the homotopy $K = -l_1^* G + l_2^* H_3$ we obtain a diagram

$$\begin{tikzcd}
(U_1 \cup U_2) \cap U_3 \arrow[r, "gl_1"] \arrow[d, "r_3"'] \arrow[dr, phantom, "\overset{\widetilde{K}}{\not\Leftarrow}"] & X^1 \arrow[d, tail, "n'"] \\
\mathbb{R}_+ \arrow[r, "\alpha_3"'] & X^2
\end{tikzcd}$$

where $n^* \widetilde{K} = K$ as tracks in $\mathcal{P}_\infty$ for the inclusion $n : X^2 \subset X$. By the PHEP we can extend the deformation H_3 in diagram (6) to a homotopy $H_3' : 1_X \not\Rightarrow p$; that is, $j_3^* H_3' = H_3$ in $\mathcal{P}_\infty$. In particular the germ p restricts to $\alpha_3 r_3$ over U_3. Similarly we can extend $\widetilde{K}$ to a homotopy $K' : n'g \not\Rightarrow g'$ over $U_1 \cup U_2$ such that $g' : U_1 \cup U_2 \to X^2$ coincides with $\alpha_3 r_3$ on $(U_1 \cup U_2) \cap U_3$. We extend g' to a germ $\tilde{g} : X \nrightarrow X^2$ by defining it as $\alpha_3 r_3$ on U_3. We have a homotopy

$$-n_* K' - G + (j_1 \cup j_2)^* H_3' : ng' \not\Rightarrow p(j_1 \cup j_2)$$

which defines, by construction, the trivial track when restricted to $(U_1 \cup U_2) \cap U_3$. By applying the PHEP there exists a new homotopy $F : ng' \not\Rightarrow p(j_1 \cup j_2)$ which is the constant homotopy $0_{\alpha_3 r_3}$ over $(U_1 \cup U_2) \cap U_3$. Therefore we can extend F to a homotopy $\widetilde{F} : n\tilde{g} \not\Rightarrow p$ by defining it as $0_{\alpha_3 r_3}$ on U_3, and finally we get the desired homotopy at ∞

$$\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "\not{\tilde{g}}"'] & \arrow[d, phantom, "\widetilde{F} - H_3' \not\Uparrow"] & X \\
& X^2 \arrow[ur, tail, "n"'] &
\end{tikzcd}$$

□

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Appendix A. Proof of Lemma 5.4

We start collecting the basic tools of pro-categories and combinatorial group theory needed for the proof of Lemma 5.4.

The *coproduct* or *wedge* $B \vee B'$ of two based sets is obtained from the disjoint union $B \sqcup B'$ by identifying both base-points. The *smash product* $B \wedge B'$ of two based sets is obtained from their product $B \times B'$ by identifying all points with a coordinate equals to the base-point. This construction is again a based set in the obvious way. We write $b \wedge b'$ for the point of $B \wedge B'$ corresponding to the pair $(b, b') \in B \times B'$.

The free group $\langle B\rangle_*$ generated by a based set B is the quotient of the free group $\langle B\rangle$ with basis B by the smallest normal subgroup containing the base-point $* \in B$. This group is isomorphic to $\langle B - \{*\}\rangle$. In fact $\langle -\rangle_*$ is a functor from the category of pointed sets to the category of groups. The right-adjoint of this functor is the forgetful functor which sends a group G to its underlying set based at $0 \in G$.

For any group G there is a natural short exact sequence

$$\langle G \wedge G'\rangle_* \overset{i}{\hookrightarrow} \langle G\rangle_* \overset{p}{\twoheadrightarrow} G, \tag{A.1}$$

where p is induced by the identity of G and i is defined by $i(a \wedge b) = [a] + [b] - [a + b]$. Here we write $[g]$ for any element $g \in G$ regarded as an element of $\langle G\rangle_*$. The naturality of i is obvious, and the exactness can be checked by using the Reidemeister–Schreier rewriting process associated to the right coset representative function $G \hookrightarrow \langle G\rangle_* : g \mapsto [g]$; see Corollary 2.7.2 and Theorem 2.9 in [19].

Functors between categories can be extended to functors between the corresponding pro-categories of towers in the obvious way. These extensions preserve adjointness relations, in particular $\langle -\rangle_*$ sends projective towers of based sets to projective towers of groups. Similarly natural transformations can be extended to pro-categories.

Lemma A.1. *An essentially monomorphic tower of based sets $\underline{B}$ with $\varprojlim \underline{B} = 0$ the trivial pointed set is a projective object.*

Proof. Assume that $\underline{B} = \{B_0 \leftarrow B_1 \leftarrow \cdots\}$ is essentially monomorphic and $\varprojlim \underline{B} = 0$. This means that (up to isomorphism) the bonding homomorphisms are inclusions of sets and $\bigcap_{i \geqslant 0} B_i = 0$.

Let

$$\begin{tikzcd} & \underline{B} \arrow[d, "h"] \\ \underline{X} \arrow[r, two heads, "f"] & \underline{Y} \end{tikzcd}$$

be a diagram of towers of based sets where f is an epimorphism. We can suppose that this diagram is given by a commutative diagram of based sets

$$\begin{tikzcd}
\cdots & X_n \arrow[l] \arrow[d, "f_n"] & X_{n+1} \arrow[l] \arrow[d, "f_{n+1}"] & \cdots \arrow[l] \\
\cdots & Y_n \arrow[l] & Y_{n+1} \arrow[l] & \cdots \arrow[l] \\
\cdots & B_n \arrow[l, hook'] \arrow[u, "h_n"] & B_{n+1} \arrow[l, hook'] \arrow[u, "h_{n+1}"] & \cdots \arrow[l, hook']
\end{tikzcd}$$

such that there is an increasing sequence k_n $(n \geqslant 0)$ with $k_n \geqslant n$ for which the image of the bonding morphism $g_n : Y_{k_n} \to Y_n$ is contained in $f_n(X_n)$; see [20], Chapter I, Section 1, Theorem 3, Chapter II, Section 2, Theorem 3 and the two last paragraphs of Chapter II, Section 2.3.

For each $x \in B_{k_n} - B_{k_{n+1}}$ we choose $\tilde{h}_n(x) \in X_n$ such that $g_n h_{k_n}(x) = f_n \tilde{h}_n(x)$. This defines a commutative diagram of based sets

$$\begin{tikzcd}
\cdots & X_n \arrow[l] & X_{n+1} \arrow[l] & \cdots \arrow[l] \\
\cdots & B_{k_n} \arrow[l, hook'] \arrow[u, "\tilde{h}_n"] & B_{k_{n+1}} \arrow[l, hook'] \arrow[u, "\tilde{h}_{n+1}"] & \cdots \arrow[l, hook']
\end{tikzcd}$$

which represents a pro-morphism $\tilde{h} : \underline{B} \to \underline{X}$ fitting into the commutative diagram

$$\begin{tikzcd}
 & \underline{B} \arrow[dl, "\tilde{h}"'] \arrow[d, "h"] \\
\underline{X} \arrow[r, two heads, "f"] & \underline{Y}
\end{tikzcd}$$

This proves that $\underline{B}$ is projective. $\square$

The converse of this lemma also holds. However, we do not include here the proof because it is not used in this paper.

Corollary A.2. *If $\underline{B}$ is a tower of based sets as in Lemma* A.1 *then $\langle \underline{B} \rangle_*$ is projective in tow-$\mathcal{G}r$*

We call the towers of groups in Corollary A.2 *generalized free towers*. They coincide with free towers as defined in Section 3 when the bonding maps of $\underline{B}$ are inclusions of subsets with finite complements.

We prove Lemma 5.4 as a consequence of the following three lemmas.

Lemma A.3. *Given an essentially monomorphic tower of groups $\underline{G}$ and a generalized free tower $\underline{L}$ there exists another generalized free tower $\underline{L}'$ together with a morphism $j : \underline{L}' \to \langle G \rangle_* \vee \underline{L}$ such that the sequence*

$$\langle \underline{G} \wedge \underline{G} \rangle_* \vee \underline{L}' \xhookrightarrow{(i,j)} \langle \underline{G} \rangle_* \vee \underline{L} \xrightarrow{(p,0)} \underline{G} \tag{A.2}$$

is exact. Here "$\vee$" denotes the coproduct in tow-$\mathcal{G}r$.

Proof. Let $\underline{B}$ be the basis pro-based set of $\underline{L}$. We can assume without loss of generality that the bonding maps of $\underline{B}$ and $\underline{G}$ are inclusions. The basis of $\underline{L}'$ is the pro-based set $\underline{B}'$ given by

$$B'_n = \{[g]b[g]^{-1};\ g \in G_n,\ b \in B_n\}$$

and the morphism j is induced by the inclusions

$$B'_n \subset \langle G_n \rangle_* \vee \langle B_n \rangle_*.$$

One readily checks that $\underline{B}'$ is a tower of inclusions with $\bigcap_{n \geqslant 0} B'_n = 0$ since $\underline{B}$ satisfies the same properties. Moreover, the exactness of (A.2) can be proved by using again the Reidemeister–Schreier rewriting processes associated to the right coset representative functions $G_n \hookrightarrow \langle G_n \rangle_* \vee \langle B_n \rangle_*$ defined by $g \mapsto [g]$ $(g \in G_n, n \geqslant 0)$; see Corollary 2.7.2 and Theorem 2.9 in [19]. □

Lemma A.4. *In the same conditions as in Lemma* 5.4 *there are free towers* $\underline{L}' = \langle \underline{B}' \rangle_*$ *and* $\underline{L}''$ *such that* $\underline{L} = \underline{L}' \vee \underline{L}''$, ϕ *is trivial on* $\underline{L}''$, *and the composite of pro-based set maps*

$$\sigma : \underline{B}' \hookrightarrow \underline{L}' \hookrightarrow \underline{L} \overset{\phi}{\twoheadrightarrow} \underline{G} \tag{A.3}$$

is a monomorphism.

Proof. Let $\underline{B}$ be the basis pro-based set of $\underline{L}$. As in the proof of Lemma A.3 we can assume that the bonding maps of $\underline{B}$ and $\underline{G}$ are inclusions. In addition, if $\psi : \underline{B} \to \underline{G}$ is the composite $\underline{B} \hookrightarrow \underline{L} \overset{\phi}{\twoheadrightarrow} \underline{G}$ we can also assume that ψ is determined by a map $\psi_0 : B_0 \to G_0$. For that map the set $\psi_0^{-1}(g)$ is finite for each $0 \neq g \in G_0$ since $\bigcap_{n \geqslant 0} G_n = 0$ and ψ is a pro-morphism. The tower $\underline{B}' \subset \underline{B}$ is obtained by choosing an element in each of the pairwise disjoint sets $\psi_0^{-1}(g)$ $(g \in G_0)$. For $g = 0 \in G_0$ we choose for $\psi_0^{-1}(0)$ the base-point of B_0 so that $\underline{B}' \subset \underline{B}$ is an inclusion of pro-based sets. The basis $\underline{B}''$ of $\underline{L}''$ is formed by the pointed sets B''_n $(n \geqslant 0)$ whose elements are either of the form $a - b$ where $a \in B_n - B'_0$, $b \in B'_0$ and $\psi_0(a) = \psi_0(b) \neq 0$ or those $a \in B_n$ with $\psi_0(a) = 0$. In particular the base point of B_n belongs to B''_n. □

Lemma A.5. *In the same conditions as in Lemma* 5.4 *the kernel of* ϕ *is a retract of a generalized free tower, and hence a projective object, in tow-*$\mathcal{G}r$.

Proof. Suppose that $\underline{L} = \underline{L}' \vee \underline{L}''$ is decomposed as in Lemma A.4. We assume that $\underline{G}$ is a tower of inclusions and that the monomorphism $\sigma : \underline{B}' \hookrightarrow \underline{G}$ in (A.3) is induced by inclusions $B'_n \subset G_n$. As towers of based sets $\underline{G} = \underline{B}' \vee \underline{G}'$ where G'_n is obtained from G_n by removing all points in B'_n except the base-point. Since $\underline{G}'$ is a projective tower of based sets and ϕ is trivial on $\underline{L}''$ there exists a map $\tau : \underline{G}' \to \underline{L}'$ such that $\phi\tau : \underline{G}' \hookrightarrow \underline{G} = \underline{B}' \vee \underline{G}'$ is the canonical inclusion. Let $\mu : \underline{G} \to \underline{L}'$ be the pro-map of pro-based sets given by $\underline{B}' \hookrightarrow \underline{L}'$

on $\underline{B}'$ and τ on $\underline{G}'$, and let $\nu : \langle \underline{G} \rangle_* \to \underline{L}'$ be the morphism in *tow-$\mathcal{G}r$* induced by μ. The diagrams

$$\begin{tikzcd} \underline{L}' \vee \underline{L}'' \arrow[r, "\phi"] \arrow[d, "\langle\sigma\rangle_*\vee 1"'] & \underline{G} \arrow[d, equal] \\ \langle \underline{G} \rangle_* \vee \underline{L}'' \arrow[r, "(p,0)"'] & \underline{G} \end{tikzcd} \qquad \begin{tikzcd} \underline{L}' \vee \underline{L}'' \arrow[r, "\phi"] & \underline{G} \arrow[d, equal] \\ \langle \underline{G} \rangle_* \vee \underline{L}'' \arrow[r, "(p,0)"'] \arrow[u, "\nu\vee 1"] & \underline{G} \end{tikzcd}$$

commute in *tow-$\mathcal{G}r$* and $\nu\langle\sigma\rangle_* = 1$, hence the kernel of ϕ is a retract of the kernel of $(p, 0)$ which is a generalized free tower by Lemma A.3. □

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